A

Synopsis

On

On Certain Aspects of Generalized Trigonometric, Hyperbolic Inequalities and their Applications

submitted to

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submitted by

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1 Introduction to proposed research topic:

In real life inequalities are more then equality. It plays vital role in almost all branches of mathematics as well as other areas of sciences like Economics, Finance, Statistics, Physics etc.

In 1934, the first book of Inequalities was written by Hardy et al[1]. The second book on this topic was written by E. Benchanbach and R. Bellman in 1961[2]. The third manuscript viz 'Analytic Inequalities' was written by D. S. Mitrinovic in 1970[3].

These books transform the field of inequalities into a systematic field and provide motivations, ideas, techniques and applications for the new research.

Since 1934, a huge amount of literature has been dedicated to the discovery of new types of inequalities and it's applications. Various inequalities have been refined and generalized by many prolific researchers. Now it is an active and attractive field of research.

In recent years, the theory of mathematical inequalities captured the attention of a large number of researchers and influenced various branches of Mathematics.

In mathematics, trigonometric and hyperbolic functions are crucial. Hyperbolic functions are analogue of the ordinary trigonometric functions but defined using the hyperbola rather than the circle.

Hyperbolic functions were introduced in the 1760's independently by Vincenzo Riccati and Johann Heinrich Lambart. The hyperbolic functions arise in many problems of mathematics and mathematical physics specially in the solutions of many linear differential equations such as equation defining a catenary.

There are some special types of inequalities related to the trigonometric and hyperbolic functions such as, the famous Wilker inequality [4], Cusa-Huygen's inequality [5], Mitrinovic inequality [3], etc.

2 Review of literature and development in the subject area:

The theory of inequalities began its development from the time when C.F. Guass, A.L.Cauchy and P.L.Cebysev, to mention only the most important, laid the theoretical foundation for approximative methods. Around the end of the 19th and the beggining of the 20th century, numerous inequalities were proved, some of which became classic, while most remained as isolated and unconnected results.

Among all inequalities, there is a number of well-known classical inequalities. Many of them have been proved by famous mathematicians and named after them. These include, in particular, Bernoulli's, Young's, Hölder's, Cauchy-Schwarz and Minkowski's inequalities[18].

Today inequalities play a significant role in all fields of mathematics, they present a very active and attractive field of research.

Few trigonometric inequality were defined initially like:

If $a \leq 3$ and $0 < x < \frac{\pi}{2}$, then

$$cosx = \left(\frac{sinx}{x}\right)^a.$$

If $a \leq b \leq 3$ and $0 < x < \frac{\pi}{2}$, then

$$cosx = \frac{(sinx)^a}{x^b}$$

This is the consequence of above result in[3].
If $0 < x < \frac{\pi}{2}$, then

$$\frac{4}{\pi}\frac{x}{\pi - 2x} < tanx.$$

There are a large number of results on inequalities involving trigonometric polynomials defined in[19].
If $0 < x < 2\pi$, then

$$-\frac{1}{2}tan\frac{x}{4} \leq \sum_{k=1}^{n} sinkx \leq \frac{1}{2}cot\frac{x}{4}.$$

Trigonometric and Hyperbolic inequalities, which have been obtained by J. Wilker [4] and C. Huygens [5] have attracted attention of several researchers[6,7,8,9,11,12,13,14].
The Wilker-type inequality for trigonometric functions is defined as
$(\frac{\sin x}{x})^2 + \frac{tanx}{x} > 2, \qquad 0 < x < \frac{\pi}{2}.$
A hyperbolic counterpart of Wilker-type inequality is defined as follows:
$(\frac{sinhx}{x})^2 + (\frac{\tanh x}{x}) > 2, \quad x \neq 0.$
has been established by L. Zhu [11].
Above two inequalities have been refined and generalized as follows:
$(\frac{\sin x}{x})^{2p} + (\frac{tanx}{x})^p > (\frac{x}{\sin x})^{2p} + (\frac{x}{\tan x})^p > 2, \qquad 0 < x < \frac{\pi}{2}.$
and
$(\frac{\sinh x}{x})^{2p} + (\frac{tanhx}{x})^p > (\frac{x}{\sinh x})^{2p} + (\frac{x}{\tanh x})^p > 2, \quad x \neq 0.$
A. Baricz and J. Sandor [14] have pointed out that, above inequalities follow from the first inequalities in:
$(cosx)^{\frac{1}{3}} < \frac{sinx}{x} < \frac{2+cosx}{3}, \qquad 0 < x < \frac{\pi}{2}.$
and

$(coshx)^{\frac{1}{3}} < \frac{sinhx}{x} < \frac{2+coshx}{3}, \quad x \neq 0.$

Here first inequality is due to D. D. Adamovic and D. S. Mitrinovic [3]. While the corresponding inequality in hyperbolic form was established by Lazarevic [7], where second inequality is due to Cusa- Huygens [5].

All these inequalities have been generalized and refined.

In this research, we are focusing on generalization of trigonometric and hyperbolic inequalities, also refinements of these inequalities.

3 Aims and Objectives of research:

1. To study different generalized trigonometric and hyperbolic functions and their important properties which can be useful for further developments.
2. To develop new useful results on generalized trigonometric and generalized hyperbolic functions.
3. To establish new inequalities involving generalized trigonometric and hyperbolic functions.
4. To refine old inequalities involving generalized elementary functions and trigonometric and hyperbolic functions.
5. To develop new method for obtaining existing and analytic inequalities.
6. To apply new results on generalized trigonometric and generalized hyperbolic functions and inequalities.
7. To explore inequalities involving *sinc* and hyperbolic *sinc* functions.
8. To study new inequalities in the set up of fuzzy sets and soft sets.
9. To develop new methods for obtaining existing and new inequalities in fractional calculus.

4 Research Methodology:

Knowledge of the concerned topic will be obtained by the survey of research papers, review articles, various books and journals. We will analyze results regarding this subject given in various journals, books. We will obtain different new results regarding generalized trigonometric and hyperbolic functions and inequalities. Light will be thrown on making graphical representation of generalized trigonometric and generalized hyperbolic functions and different bounds of generalized trigonometric and hyperbolic functions. Also we would like to visit different research institutes like IIT's, TIFR, SGB Amravati University, Amravati, IISc etc, to interact with eminent mathematicians, also would like to attend national, international conferences related to our research work.

5 Importance of study:

We hope that, the work of this research will be useful for the engineers, scientists and mathematicians, researchers at the international, national level. Basic aim of our study is to generalize and sharpen the inequalities involving the generalized trigonometric, generalized hyperbolic, *sinc* function, generalized logarithmic, exponential functions etc. There is a large scope to investigate new inequalities involving generalized trigonometric and hyperbolic functions. The present work will be useful for budding researchers.

International Status:

Many mathematicians are working in this field. Some of them are J.Sándor (Babeș-Bolyai University, Cluj-Napoca, Romania), Edward Neuman (Professor Emeritus of Mathematics, Southern Illinois University Carbondale), Tuo Leng (School of Computer Engineering and Science, Shanghai University), Marko Kostić (University of Novi Sad), Branko Malesevic (University of Belgrade), Christophe Chesneau (University of Caen Normandie, France), Bojan Banjac (University of Novi Sad), Horst Alzer (Waldbrol), Man Kam Kwong (Hong Kong), Barkat A. Bhayo (IBA University, Sindh, Pakistan), Xiao-Diao Chen (Hangzhou Dianzi University), etc.

National Status:

Some of the Indian mathematicians are also working on same lines. Few of them are DB Pachpatte (Dr. Babasaheb Ambedkar Marathwada University, Aurangabad).

6 Plan of work:

Sr.No.	Period	Plan of the work	Targets to be Achieved
1	First year	Study of the background required for the research topic.	Good command over the subject and clear the course work.
2	Second year	Collecting the papers and analyze them.	Try to get some construction techniques about inequalities.
3	Third year	Work on some problems to be included in the thesis.	Submit at least two research papers.
4	Fourth year	Publishing some papers and submitting the thesis.	Some more papers and thesis submission.

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